Excitation Fronts on a Periodically Modulated Curved Surface

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The evolution of an excitation front propagating on a nonuniformly curved surface is considered within the framework of a kinematical model of its motion. For the case of a surface with a periodically modulated curvature an exact solution of the front shape is obtained under the assumption of sufficiently small surface deformation. The results of the theoretical consideration are compared with the experimental data obtained with a modified Belousov-Zhabotinsky reaction in a thin nonuniformly curved layer.

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Excitation fronts and waves are observed in many biological systems such as cardiac muscle tissue [1] or aggregating slime mold cells [2], physical systems such as CO oxidation on platinum surfaces [3], and chemical systems such as the Belousov-Zhabotinsky (BZ) reaction [4–6]. The properties of these waves differ considerably from the acoustic or electromagnetic waves in traditional conservative systems. This difference is due to the active local dynamics of excitable media.

The investigation of spatiotemporal structures in excitable media has attracted increasing interest during the last decades. So far, most of the theoretical and experimental studies of two-dimensional excitable systems were dealing with homogeneous and nonhomogeneous distribution of different control parameters in planar geometries [7–9]. In natural systems, however, many excitable reactions occur on curved surfaces, e.g., on the heart muscle [1] or in the phenomenon of spreading depression waves in chicken retina [10]. Curvature can essentially affect the wave processes, as first discussed in [11] and later investigated in [12–18].

In this Letter, we report how an excitation front evolves on a nonuniformly curved surface. As a generic case of a curvature geometry we start to consider a periodically modulated surface. The role of such a perturbation is investigated analytically, numerically, and by experimental observations.

In order to simulate wave propagation through an excitable medium a general two-component reaction-diffusion system is widely used:

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_u \nabla^2 u + F(u,v)\,,\\ \frac{\partial v}{\partial t} &= D_v \nabla^2 v + \epsilon G(u,v)\,,\end{aligned} \tag{1}$$

where the activator (u) and the inhibitor (v) may represent concentrations of chemical species, temperature, or electric potential, etc. The nonlinear functions $F(u,v)$ and $G(u,v)$ describe the excitable local kinetics of the system. The transport processes in this medium are due to diffusion of activator and inhibitor with diffusion constants D_u and D_v, respectively.

It is known that the propagation velocity V of an excitation front on a plane depends linearly on its local curvature K [19,20] according to

$$V = V_0 - DK\,. \tag{2}$$

V_0 is the velocity of a planar wave and the slope D is approximately equal to the diffusion constant D_u. A generalization of this equation for a curved surface results in the same equation, but the value K in this case is the geodetic curvature of the front. Note that the geodetic curvature of a curve on an arbitrary surface is the curvature of its projection on the tangential plane. The geodetic curvature K given as a function of the arc length $K = K(l,t)$ specifies the shape of the front on a surface at a given time instant, like a natural equation of a curve does on a plane.

It has been shown [10,13] that the evolution of the front shape $K(l,t)$ obeys the following integrodifferential equation:

$$\frac{\partial}{\partial l}\left[K\int_0^l K(\xi,t)V(\xi,t)\,d\xi\right] + \frac{\partial K}{\partial t} + \frac{\partial^2 V}{\partial l^2} = -\Gamma V\,, \tag{3}$$

where Γ is the local Gaussian curvature of the surface. If the shape of the surface is written as

$$z = f(x,y)\,, \tag{4}$$

the value of the Gaussian curvature can be represented by the following expression:

$$\Gamma(x,y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}\,. \tag{5}$$

Let us consider the evolution of an initially flat front on a surface with slightly modulated Gaussian curvature:

$$z = A\sin(bx)\sin(by)\,. \tag{6}$$

This surface represents the easiest periodically nonuniformly curved surface. An example of such a surface is shown in Fig. 1. The condition of a small modulation can be written as $Ab \ll 1$. Under this condition the Gaussian curvature of the surface (6) writes

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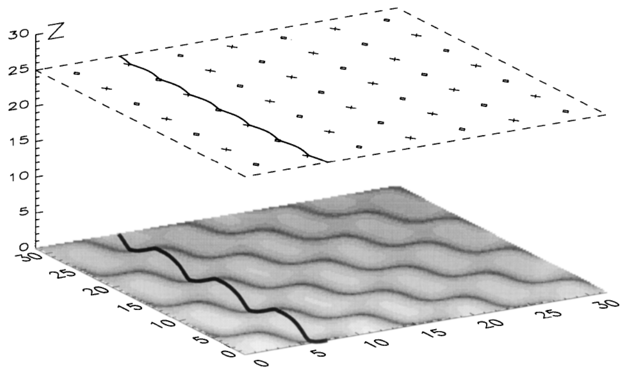

FIG. 1. Excitation front (thick solid line) propagating on a nonuniformly curved surface [Eq. (6)] with $A = 1.0$ and $b = 0.7$. The solid line in the upper part represents the projection of the front on a plane $z = \text{const}$. The diamonds and crosses indicate local maxima and minima of the nonuniformly curved surface, respectively.

$$\Gamma(x,y) = A^2b^4(\sin^2(bx)\sin^2(by) - \cos^2(bx)\cos^2(by)) = -\frac{1}{2}A^2b^4(\cos(2bx)\cos(2by)). \quad (7)$$

It follows from (3) that a modulation of the Gaussian curvature will induce deformations of the initially flat front. In a linear approximation these deformations can be considered as sufficiently small: $K \ll b \ll V_0/D$. Keeping only linear terms in K, the basic kinematical Eq. (3) can be reduced to the following form:

$$\frac{\partial K}{\partial t} - D\frac{\partial^2 K}{\partial l^2} = -\Gamma V_0. \quad (8)$$

Let α be the angle between the front and the x axis. In a planar region the wave propagates with V_0. On a surface with small deformations, the front should propagate with velocity $V \simeq V_0$. In a comoving coordinate system with a constant velocity, the Gaussian curvature along the front can be expressed as a function of the arc length l and time t according to

$$\Gamma(l,t) = -\Gamma_0(\cos(\beta_1 l - \omega_1 t) + \cos(\beta_2 l + \omega_2 t)), \quad (9)$$

where $\Gamma_0 = A^2b^4/2$, $\beta_1 = 2b\cos\alpha$, $\omega_1 = 2bV_0\sin\alpha$, $\beta_2 = 2b\sin\alpha$, $\omega_2 = 2bV_0\cos\alpha$. This expression is substituted into (8), which in this case can be solved analytically by use of Fourier transform techniques. One obtains the expression

$$K = \frac{A^2b^4V_0}{2}\left[\frac{\cos(2bl\cos\alpha - 2bV_0t\sin\alpha + \theta_1)}{(16D^2b^4\cos^4\alpha + 4b^2V_0^2\sin^2\alpha)^{1/2}} + \frac{\cos(2bl\sin\alpha + 2bV_0t\cos\alpha - \theta_1)}{(16D^2b^4\sin^4\alpha + 4b^2V_0^2\cos^2\alpha)^{1/2}}\right], \quad (10)$$

with $\tan\theta_1 = V_0\sin\alpha/(2Db\cos^2\alpha)$ and $\tan\theta_2 = V_0\cos\alpha/(2Db\sin^2\alpha)$.

The evolution of the wave shape $K(l,t)$ is described as a superposition of two waves propagating in opposite directions. This interplay strongly depends on the angle α, as can be illustrated by considering two particular cases.

Let us choose $\alpha = 0$, which corresponds to the wave propagating along the y axis. Then we have $\beta_1 = 2b$, $\omega_1 = 0$, $\beta_2 = 0$, $\omega_2 = 2bV_0$, $\theta_1 = 0$, $\theta_2 = \pi/2$, and the general solution (10) reduces to

$$K = \frac{A^2b^2V_0}{8D}\left[\cos(2bl) + \frac{2bD}{V_0}\sin(2bV_0t)\right]. \quad (11)$$

One term in this equation describes spatial and the other one temporal oscillations of the front shape. By averaging the temporal oscillations, a function $K(l)$ is obtained, which has a period twice shorter than the spatial period of the surface (6) in x direction. Note that if the estimate $2bD/V_0 \ll 1$ is valid, the amplitude of the temporal oscillations around this average shape remains small. All curves in Fig. 2(a) are superimposed at the last position of the front to show the oscillations of the front shape.

For a front propagating in diagonal direction (e.g., $\alpha = \pi/4$), the general solution is characterized by the following values of the coefficients: $\beta_1 = \beta_2 = \sqrt{2}\,b$, $\omega_1 = \omega_2 = \sqrt{2}\,bV_0$, $\tan\Theta_1 = \tan\Theta_2 = V_0/(\sqrt{2}\,Db)$. Now the solution can be written as

$$K = \frac{A^2b^3V_0}{(4D^2b^2 + 2V_0^2)^{1/2}}\cos(\beta_1 l)\cos(\omega_1 t - \Theta_1). \quad (12)$$

This expression can be interpreted as a "standing wave" in that some ripples appear periodically on the initially flat front at some well-determined places as shown in Fig. 2(b). The spatial period of the ripples is larger than in the previous case, and their amplitude should be considerably smaller, if the inequality $2bD/V_0 \ll 1$ holds.

A novel experimental methodology has been developed to allow the experimental investigation of front propagation on nonuniformly curved surfaces. A computer

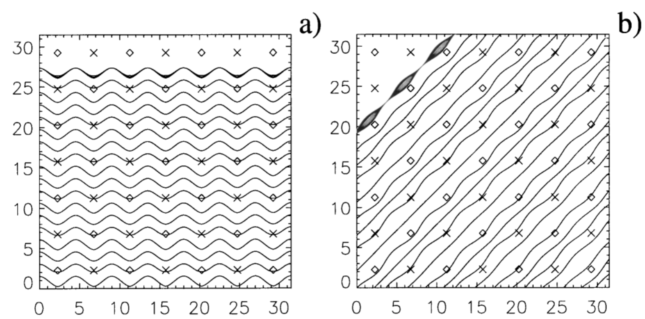

FIG. 2. Temporal sequence of the momentary position and form of the wave front with $\Delta t = 40$ s (thin solid lines) corresponding to solutions (11) and (12) of the kinematical Eq. (8) obtained for (a) wave propagating along the y axis ($\alpha = 0$) and (b) in diagonal direction ($\alpha = \pi/4$) with $V_0 = 0.038$ mm/s and $D = 1.9 \times 10^{-3}$ mm^2/s. These curves are superimposed at the last position of the front to show the deformations of the front shape. The diamonds and crosses indicate local maxima and minima of the nonuniformly curved surface, respectively.

numerical control (CNC) machine is used to manufacture pairs of complementary Plexiglas molds that, once put together, create a thin hollow space, which can accommodate a chemical reaction system. This quadratic space has a side length of 61.50 mm and a height of 0.40 mm. Centered within this square is a $(31.50 \times 31.50)\ \mathrm{mm}^2$ area that defines the nonuniformly curved surface. For the experiments presented here, the surface is given by the two functions $h_1(x, y)$ and $h_2(x', y')$ describing the surface undulations of the two Plexiglas molds within this area:

$$\begin{aligned} h_1(x, y) &= A \sin(bx) \sin(by), \\ h_2(x', y') &= A \sin(bx') \sin(by') + \delta/(Ab\chi), \end{aligned} \tag{13}$$

with

$$\begin{aligned} x' &= x - [\delta \cos(bx) \sin(by)]/\chi, \\ y' &= y - [\delta \sin(bx) \cos(by)]/\chi, \\ \chi &= [1/2 - (1/2)\cos(2bx)\cos(2by) + 1/(Ab)^2]^{1/2}, \end{aligned}$$

and $A = 1.00$ mm, $b = 2\pi/9.00\ \mathrm{mm}^{-1}$. This choice of functions establishes a constant distance $\delta = 0.40$ mm between the two surfaces in normal direction.

The hollow space created by the molds is filled with a ferroin-catalyzed Belousov-Zhabotinsky solution. Malonic acid, the classical organic substrate of this reaction, is replaced by 1,4-cyclohexanedione (CHD) to prevent the formation of undesired CO_2 bubbles [21]. Excitable behavior is observed for the following set of initial concentrations, which have been used in this study: [CHD] = 0.12 M, $[NaBrO_3] = 0.08$ M, $[H_2SO_4] = 1.50$ M, and [ferroin] = 0.6 mM. The waves were initiated from a straight silver wire that was placed between the molds. Note that the reaction solution was filled into the mold approximately 50 min after preparation to minimize the number of spontaneous pacemakers. Wave propagation was detected with a charge-coupled-device (CCD) camera and the resulting video signal was digitized using a PC-based image-acquisition board (640×480 pixels) for further analysis.

Within the framework of the presented kinematical theory, experiments with initially either parallel ($\alpha = 0$) or diagonal ($\alpha = \pi/4$) planar waves were analyzed explicitly and are presented in Figs. 3 and 4.

Figure 3(a) shows consecutive snapshots of the front of an initially planar excitation wave propagating in horizontal direction from the left to the right-hand side across the surface (time interval between snapshots, 20.0 s). The thick line indicates the boundary between the planar (left) and the nonuniformly curved region (right). The position of its local maxima and minima are represented by diamonds and crosses, respectively. The front shows significant deviations from the initially planar geometry.

The part of the waves which propagates over the nodes is always leading, whereas the front always lags behind at the location of the extrema. The experimental data reveal the main phenomena predicted by Eq. (11) and shown in

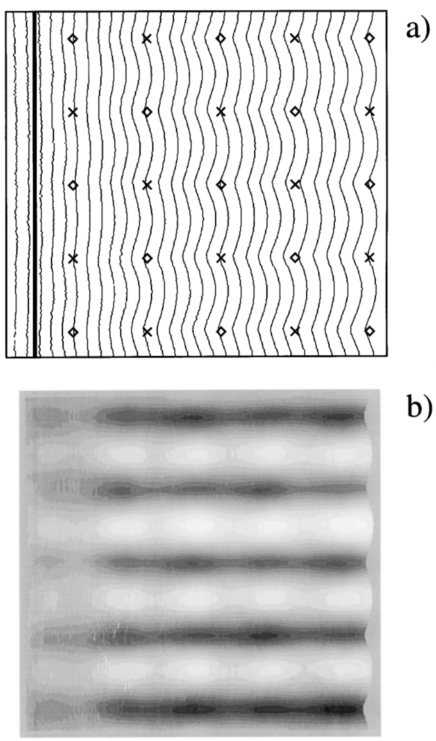

FIG. 3. (a) Superposition of 31 experimental snapshots of an excitation front propagating in a curved Belousov-Zhabotinsky system. The front travels in a horizontal direction and undergoes systematic front deformations. The thick line indicates the boundary of the nonuniformly curved surface. Time between snapshots: $\Delta t = 20$ s. Image size: $(28 \times 22)\ \mathrm{mm}^2$. (b) Deformation map obtained from the experiment shown in (a). Dark (bright) areas correspond to regions in which the arrival of the front is delayed (early) with respect to the time expected for a planar front. Local values within the map were calculated from the distance between the actual front and a median line. This calculation was repeated successively at 1 s intervals for 600 s. Image area: $(26 \times 22)\ \mathrm{mm}^2$.

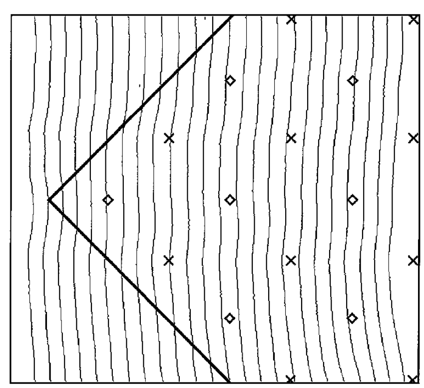

FIG. 4. Superposition of 25 experimental snapshots of an excitation front propagating in diagonal direction across a curved Belousov-Zhabotinsky system. Thick lines and spots indicate the boundary and the local extrema of the nonuniformly curved surface, respectively. Time between snapshots: $\Delta t = 20$ s. Image size: $(26 \times 20)\ \mathrm{mm}^2$.

Fig. 2(a). This particular finding is further illustrated in Fig. 3(b). Since the long scale geometry of the evolving front remains planar, we analyzed its periodic deformations by calculating the shortest distance d between every point of the actual front and its medium value. The value of d is negative, if the local front segment is retarded with respect to the fitted line and positive in the opposite case. The gray levels in Fig. 3(b) represent the values of d as obtained from 600 consecutive snapshots of the propagating front. They vary between approximately -0.52 and 0.47 mm. The figure reveals five dark stripes in the horizontal direction, each centered along a line which connects neighboring extrema of the surface. These stripes are separated by bright regions, which correspond to areas in which the front has a convex shape when moving across the nodes of the surface.

The deformations for a wave propagating in diagonal direction across the surface are shown in Fig. 4 with 31 consecutive front lines, separated each by a time interval of 20.0 s. In this case the deformations are smaller than in Fig. 3(a). This is in good agreement with the prediction of Eq. (12) as discussed above.

The theoretical and experimental studies performed in this Letter describe wave evolution on a periodically modulated curved surface as a complicated spatiotemporal process. However, for the particular cases of $\alpha = 0$ and $\alpha = \pi/4$ two unexpected regimes are observed which allow a rather simple description [see Eqs. (11) and (12)]. The close correspondence between the theoretical and experimental results demonstrates that some generic features of the wave dynamics on nonuniformly curved surfaces have been discovered. The experimental method developed here can be applied to construct quite different surface shapes (e.g., semispheres, paraboloids), which could be interesting for further studies on curvature effects. It should be emphasized that a similar theoretical approach can be also applied to a variety of systems, in which velocity changes arise from sources other that curvature, i.e., photochemical perturbations, temperature and concentration variations, or ordered heterogeneities of the active medium [22–24].

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